

# Planar Approximation as Two-Field Boltzmann Theory

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## Abstract

A modified interaction representation for the master field describing connected  $SU(N)$ -invariant Wightman's functions in the large  $N$  limit of matrix fields is constructed. This construction is based on the representation of the master field in terms of Boltzmannian field theory found before [1]. In the modified interaction representation we deal with two scalar Boltzmann fields (*up* and *down* fields). For up and down fields only half-planar diagrams contribute and this could help to write down a recursive set of non-linear integral-differential equations summing up planar diagrams.

Recently the master field describing the large  $N$  limit in matrix models and QCD in four dimensional space-time has been constructed [1]. The obtained master field satisfies to the ordinary equations of relativistic field theory but fields are quantized according to a new rule in so called Boltzmannian Fock space. Boltzmann field acting in the free (Boltzmannian) Fock space was used by Gopakumar and Gross [2] and Douglas [3] in construction of the master field for 0-dimensional matrix models (for discussion of the Boltzmannian Fock space see also [5] and refs. therein). Boltzmann field was also used in construction of the master field for QCD<sub>2</sub> [4]. About definition of the master field and early attempts of construction of the master field see refs. in [1, 7].

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The construction [1] of master field is based on the Yang-Feldman equations and it deals not with Green's functions but with the Wightman's correlation functions. As it is well known in the case of Bose or Fermi fields the Yang-Feldman formalism is equivalent to the standard Feynman-Dyson diagram technique for  $T$ -product and one has the standard  $T$ -product expression for S-matrix in interaction representation [11]. In the Boltzmann field theory we have a more general field algebra than ordinary Bose or Fermi algebras and as a result one cannot use the standard interaction picture. An appropriated interaction representation for half planar diagrams, which form a subclass of planar diagrams, has been constructed in [6] (see also this volume).

The goal of this talk is to present an interaction representation for the master field obtained in [1]. This new representation is more convenient for study of some aspects of the master field than the Yang-Feldman formalism. In particular, this representation seems more suitable for investigations of renormalizability <sup>1</sup> and gauge invariance <sup>2</sup>. Moreover, we hope that the existence of closed set of equations for the lowest Green's function for the simplest interaction representation for Boltzmann theory permits to write a tractable non-perturbative recursive set of non-linear integral-differential equations summing up planar diagrams. This subject requires a further investigation.

Let us remain the main result of [1]. There it has been considered  $U(N)$ -invariant Wightman's functions in the Yang-Feldman formalism,

$$W(x_1, \dots, x_k) = \frac{1}{N^{1+\frac{k}{2}}} < 0 | \text{tr} (M(x_1) \dots M(x_k)) | 0 > \quad (1)$$

where  $M(x) = (M_{ij}(x))$ ,  $i, j = 1, \dots, N$  is an Hermitian scalar matrix field in the 4-dimensional Minkowski space-time with the field equations

$$(\square + m^2)M(x) = J(x), \quad J(x) = - \sum \frac{c_k}{N^{-1+k/2}} M^{k-1}(x) \quad (2)$$

where  $c_k$  are the coupling constants which have no a dependence on  $N$ . One integrates eq (2) to get the Yang-Feldman equation [10, 11]

$$M(x) = M^{(in)}(x) + \int D^{ret}(x-y) J(y) dy \quad (3)$$

where  $D^{ret}(x)$  is the retarded Green function for the Klein-Gordon equation,

$$D^{ret}(x) = \frac{1}{(2\pi)^4} \int \frac{e^{-ikx}}{m^2 - k^2 - i\epsilon k^0} dk$$

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<sup>1</sup>Renormalizability in Yang-Feldman formalism for Bose systems was investigated by Dyson [12] and Khallen [13]

<sup>2</sup>Gauge invariance of the planar approximation for Yang-Mills theory has been studied recently in [14]

and  $M^{(in)}(x)$  is a free Bose field

$$[M_{kl}^{(in)}(x), M_{l'k'}^{(in)}(y)]|_{x_0=y_0} = \delta_{k,k'}\delta_{l,l'}D^-(x-y) \quad (4)$$

$|0\rangle$  is the Fock vacuum for the free field  $M^{(in)}(x)$ .

In [1] it has been show that the limit of functions (1) when  $N \rightarrow \infty$  can be expressed in terms of a quantum field  $\phi(x)$  (the master field) which is a solution of the equation

$$\phi(x) = \phi^{(in)}(x) + \int D^{ret}(x-y)j(y)dy, \quad j(x) = -\sum c_k \phi^{k-1}(x) \quad (5)$$

The field  $\phi(x)$  does not have matrix indexes. The free scalar Boltzmannian field  $\phi^{(in)}(x)$  is given by

$$\phi^{(in)}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega(k)}} (a^*(k)e^{ikx} + a(k)e^{-ikx}), \quad (6)$$

where  $\omega(k) = \sqrt{k^2 + m^2}$ . It satisfies to the Klein-Gordon equation

$$(\square + m^2)\phi^{(in)}(x) = 0$$

and it is an operator in the Boltzmannian Fock space with relations

$$a(k)a^*(k') = \delta^{(3)}(k-k'), \quad \text{and} \quad a(k)|\Omega_0\rangle = 0 \quad (7)$$

The relation between the large N invariant Wightman's functions and the Boltzmannian Wightman functions is the following

$$\lim_{N \rightarrow \infty} \frac{1}{N^{1+\frac{k}{2}}} < 0 | \text{tr} (M(x_1) \dots M(x_k)) | 0 \rangle = (\Omega_0 | \phi(x_1) \dots \phi(x_k) | \Omega_0) \quad (8)$$

where the field  $M(x)$  is defined by (3) and (4) and  $\phi(x)$  is defined by (5),(6) and (7).

In this talk I am going to show that for the master field exists the following interaction representation for connected Wightman's functions

$$(\Omega_0 | \phi(x_1) \dots \phi(x_k) | \Omega_0) = (0 | \phi_d(x_k) \dots \phi_d(x_1) \frac{1}{1 + \int d^Dx \mathcal{L}(\bar{\phi}_d(x), \phi_d(x), \phi_u(x))} | 0) \quad (9)$$

Fields  $\bar{\phi}_d(x)$ ,  $\phi_d(x)$  and  $\phi_u(x)$  are Boltzmann fields. The first two fields we call the down fields and  $\phi_u(x)$  field we call the up field. The meaning of this terminology will be clear below. The up and down fields are mutually commute and they are sums of creation and annihilation parts

$$\phi_d(x) = \phi_d^+(x) + \phi_d^-(x), \quad \bar{\phi}_d(x) = \bar{\phi}_d^+(x) + \bar{\phi}_d^-(x), \quad \phi_u(x) = \phi_u^+(x) + \phi_u^-(x), \quad (10)$$

$$\phi_u^-(x)|0\rangle = \phi_d^-(x)|0\rangle = \bar{\phi}_u^-(x)|0\rangle = 0 \quad (11)$$

$$(0|\phi_u^+(x) = (0|\phi_d^+(x) = (0|\bar{\phi}_u^+(x) = 0, \quad (12)$$

and they satisfy the following operator algebra

$$\bar{\phi}_d^-(x)\phi_d^+(y) = D^{ret}(x-y), \quad \phi_d^-(y)\bar{\phi}_d^+(x) = D^{ret}(x-y), \quad (13)$$

$$\phi_u^-(x)\phi_u^+(y) = D^-(y-x), \quad \phi_d^-(x)\phi_d^+(y) = \bar{\phi}_d^-(x)\bar{\phi}_d^+(y) = 0 \quad (14)$$

This algebra has a realization in the tensor product of  $\mathcal{H}_u$  and  $\mathcal{B}_d$  spaces.  $\mathcal{H}_u$  is the Boltzmannian Fock space.  $\mathcal{B}_d$  is a linear space with a bilinear form. This linear space is spanned on linear combinations of products of creation part of down fields applying to the vacuum  $|0\rangle$

$$(\phi_d^+(y_1))^{k_1}(\bar{\phi}_d^+(z_1)^{p_1} \dots (\phi_d^+(y_l))^{k_l}(\bar{\phi}_d^+(z_l)^{p_l}|0\rangle).$$

Algebra (13),(14) is invariant under the following  $*$  operation

$$(\phi_d^-(x))^* = \phi_d^+(y), \quad (\bar{\phi}_d^-(x))^* = \bar{\phi}_d^+(y), \quad (\phi_u^-(x))^* = \phi_u^+(y), \quad (15)$$

Lagrangian  $\mathcal{L}$  in formula (9) is defined by the form of the matrix field Lagrangian. For example, for the simplest case of the cubic interaction

$$L = \frac{1}{2} \text{tr} (\partial_\mu M)^2 + \frac{1}{2} m^2 \text{tr} (M)^2 + \frac{g}{3 \cdot N^{1/2}} \text{tr} M^3, \quad (16)$$

$\mathcal{L}$  has a form

$$\mathcal{L} = ((\square + m^2)\bar{\phi}_d(x))\phi_u + g[\bar{\phi}_d(x)(\phi_d^2(x) + \phi_d(x)\phi_u(x) + \phi_u^2(x)) + \phi_u(x)\phi_d(x)\bar{\phi}_d(x)] \quad (17)$$

Note that  $\mathcal{L}$  is not invariant under  $*$ -operation. This is not surprising if one takes into account that the construction of the master field in the 0-dimensional case also deals with the non-Hermitial master field [2].

Together with (9) the representation (8) gives

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N^{1+\frac{k}{2}}} &< 0 | \text{tr} (M(x_1) \dots M(x_k)) | 0 > \\ &= (0 | \phi_d(x_k) \dots \phi_d(x_1) \frac{1}{1 + \int d^D x \mathcal{L}(\bar{\phi}_d(x), \phi_d(x), \phi_u(x))} | 0) \end{aligned} \quad (18)$$

An analytical origin of the representation (18) is the representation (8). A topological origin of this representations is that any planar diagram can be redrawn so that all vertexes lie on some straight line dividing the plane on which diagram is drawn on two half planes, the upper and down half plane. All lines in respect of this division of the plane are divided on upper and down lines. Upper and down lines have no intersections. There are many possibilities to perform a decomposition of the lines of the planar diagram into upper and down lines. We will prove that the special

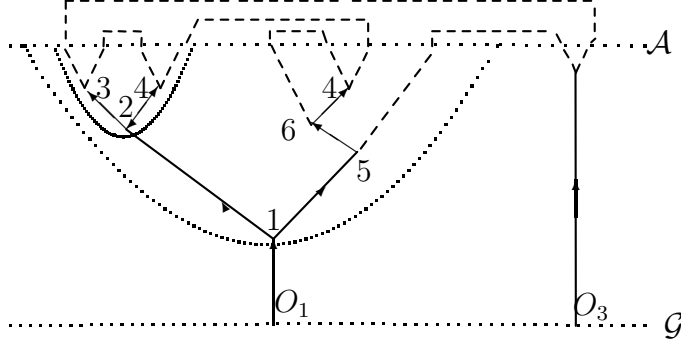


Figure 1: Diagram representing two-point Wightman's function

form (9) with relations (10)-(14) solve the corresponding combinatorial problem in the proper way.

To this end let us first make few comments about an interpretation of the RHS of relation (8) on diagrams. For simplicity we consider the case of the cubic interaction. A perturbation expansion on the coupling constant  $c_3 \equiv g$  of the RHS of (8) on diagrams language can be represented in the following way (see Fig.1). A diagram representing a  $k$ -point Wightman's function contains  $k$  trees "planted" on the same "ground" and the trees have no overlapping. On Fig.1  $\mathcal{G}$ -line represents the "ground" line and there are two trees "planted" on the line  $\mathcal{G}$ . Each tree represents a term of the perturbative solution of operator equation (5). Trees contain vertexes and links. There are two type of links. We call them branches and leaves, respectively. A branch relates two vertexes and a leaf is attached only to one vertex. A branch of the tree represents  $D^{ret}(y_{i,j} - y_{i+1,j})$ , where  $y_{i,j}$  and  $y_{i+1,j}$  are coordinates of  $i$ -vertex and  $i + 1$ -vertex belong  $j$ -tree, respectively. We draw branches by solid lines. Branches have direction. Each vertex of the tree contains exactly one branch coming in and several branches coming out. Trees contain also several leaves which are attached to some vertexes. Leaves represent *in*-fields. We draw leaves by dash lines. So there are several types of vertexes in the tree representing the perturbative solution of (5) for the given interaction. For the cubic interaction there are four types of vertexes (see Fig.2). Since equation (5) is operator valued equation and operators  $\phi^{in}$  do not commute it is essential to preserve in the perturbation solution the operator order of  $\phi^{in}$ . To keep the operator ordering we draw the tree so that its branches have no overlapping. All *in*-fields which are ordered according formula (5) are also ordered in the diagram. To make the ordering more obvious one can deform the tree so that all vertex containing leaves are drawn on some auxiliary line (line  $\mathcal{A}$  on Fig.1 and Fig.3a,b).

Note that the same picture is true also for diagrams representing the solution of

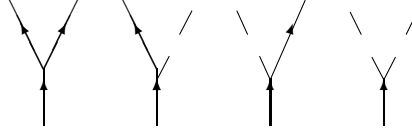


Figure 2: Types of vertexes in diagrams representing Wightman's functions in the Yang-Feldman formalism for the cubic interaction

the Yang-Feldman equation for the usual Bose field. The difference comes out then we consider expectation values of products of several fields being solutions of the Yang-Feldman equation. For example, the diagram Fig.3a also describes the operator product of two Bose scalar fields being solutions of the Yang-Feldman equation.

Since we consider expectation values in respect of the *in*-vacuum all leaves should be contracted and since we are doing calculations in the Boltzmannian case the contractions of leaves should have no intersections with themselves. By our construction leaves also have no intersections with branches. So, leaves are contracted in pairs without intersections with others leaves or branches. We draw contractions of leaves by dash lines. For the Bose case all contractions of leaves are admissible. For example, the diagram Fig.1 describes the two-point Wightman's function of the scalar field theory with the cubic interaction, but together with this diagram diagrams with overlapping contractions of leaves also contribute.

As we just have mentioned we can deform the given diagram so that it exists an auxiliary line which crosses all leaves (line  $\mathcal{A}$  on Fig.1). After that we can redraw the diagram so that all leaves are started from this auxiliary line. In this new picture (Fig.3b) all vertexes containing at least one leaf are on the auxiliary line  $\mathcal{A}$ . To find a point on the auxiliary line corresponding to a vertex with only one leaf it is enough to take a intersection of the corresponding dash line with the auxiliary line. If a vertex contains two leaves one can take as a image of the corresponding vertex the intersection of the auxiliary line with the left leaf. We note that after such redrawn contractions of the leaves are represented as a set of top "arcs" based on the auxiliary line.

Moreover, we can redraw each of diagrams representing (8) so that also all other vertexes as well as the beginning of the external lines are drawn on the same auxiliary line  $\mathcal{A}$  (see Fig.3c). To make this fact obvious let us note that any tree  $\mathcal{T}_{0,j}$  is dual to a set of bottom arcs [8] so that to each part  $\mathcal{T}_{i,j}$  of tree  $\mathcal{T}_{0,j}$  corresponds an arc which surrounds this tree  $\mathcal{T}_{i,j}$ , ( $\mathcal{T}_{i,j}$  is itself a tree;  $i$  specified the  $i$ -vertex of the  $j$ -tree, see Fig.3a).

Moving each vertex  $\mathcal{V}_{i,j}$  to some point belong a left segment of the auxiliary line  $\mathcal{A}$  which is cut out by the arcs dual to the nearest trees we see that all vertexes of diagram are now on the auxiliary line  $\mathcal{A}$ . For example, the image of vertex 1 on 3b

must be on the segment  $(g_1 g_2)$ . Points  $1'$  and  $2'$  on Fig.3c are images of vertexes 1 and 2. Images of "ground" points (points  $O'_1$  and  $O'_3$  on Fig.3c) are also on the auxiliary line. All branches of the given tree are drawn now as solid bottom arcs and these arcs are started from the images of "ground" points or vertexes on the auxiliary line. According this construction the solid arcs have no overlapping. Some vertex are also bases for dash arcs which represent contractions of the leaves, or *in*-fields. Note that after this redraw all external lines are also started from the point lying on the auxiliary line  $\mathcal{A}$  and they are arranged in respect of the original ordering in the ground line in the opposite way. Since a solid line comes in each vertex and all vertex are situated in the right of external points we can reconstruct a part of the diagram that is represented by dash lines as the result of the calculation of the following expectation value in the Boltzmann theory

$$\langle 0 | \phi_d(x_1) \dots \phi_d(x_l) \dots \bar{\phi}_d(y_i) (\phi_d(y_i))^{k_i} \dots \phi_d(y_l) \bar{\phi}_d(y_l) \dots | 0 \rangle, \quad (19)$$

$k_i = 0, 1, 2$   $i, j = 1, \dots, n$ ,  $i \neq j$ . We see that there are four types of vertexes. A vertex with only one down field corresponds to points  $3'$  and  $4'$  on Fig.3c. Vertex  $\bar{\phi}_d \phi_d$  corresponds to point  $6'$  and vertex  $\phi_d \bar{\phi}_d$  corresponds to point  $5'$ . Vertex  $\bar{\phi}_d \phi_d^2$  corresponds to points  $1'$  and  $2'$ .

An analytical expression which corresponds to contractions of leaves comes out if we introduce the Boltzmannian field  $\phi_u$  which has a decomposition (10) with algebraic relations (13),(14). Since upper and down lines do not "observe" each others it is natural to assume that the corresponding fields are mutually commuted.

For some  $\phi(x)$  we can take just a simplest approximation  $\phi(x) = \phi^{in}(x)$  and therefore leaves may be attached to some external points. Note that the ordering of this field in respect to others *in*-fields attached to vertex lying on the auxiliary line is essential. To keep properly this ordering we add to  $\mathcal{L}$  the quadratic term  $((\square + m^2) \bar{\phi}_d(x) \phi_u, \bar{\phi}_d(x)$  being contracted with the field  $\phi_d(x)$  after applying  $\square + m^2$  produces  $\delta(x - x_i)$  and gives  $\phi_u$  which is in the right place (see Fig.4). This remark completes the prove of the representation (9).

In conclusion, we have shown that in the large  $N$  limit there exists the new "interaction" representation for connected  $SU(N)$ -invariant Wightman's functions of matrix fields in term of scalar down and up fields  $\bar{\phi}_d(x)$ ,  $\phi_d(x)$  and  $\phi_u(x)$ . Note that the new interaction representation contains a rational function of the interaction Lagrangian instead of the exponential function in the standard interaction representation. Selecting a part of terms in this interacting representation we get so-called half-planar diagrams. The Schwinger-Dyson equations for the 2- and 4-point half-planar correlation functions for this theory form a closed system of integral equations. More rich set of terms in the representation (9) reproduces a parquet approximation (generalized ladder diagrams) in matrix models [15]. By means of numerical calculations it has been demonstrated that in the large  $N$  limit the parquet approximation for 0-dimensional models gives a good agreement with exact results.

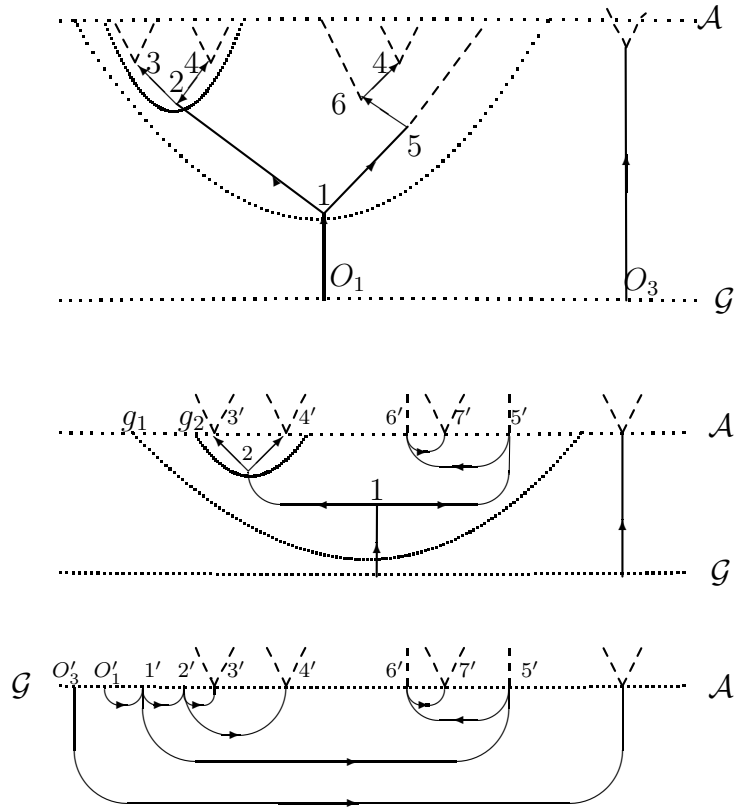


Figure 3: Trees contributing to two-point Wightman's function



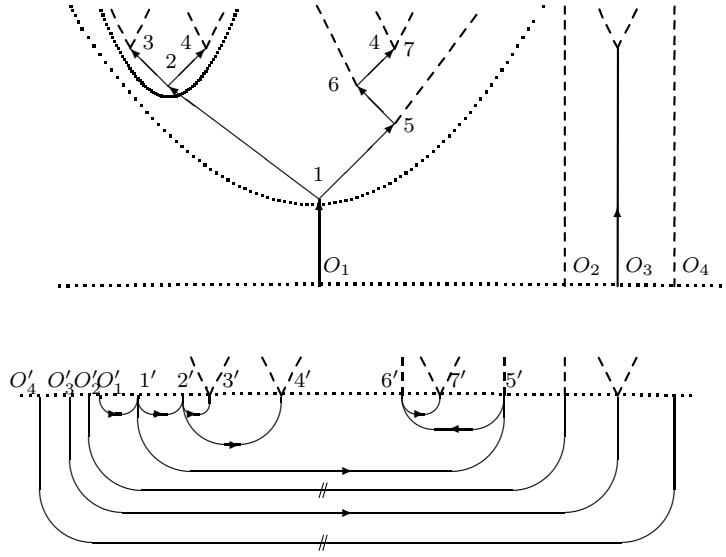


Figure 4: Trees contributing to four-point Wightman's function

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